1 Calculus review

We will often use nonlinear curves for examples in this course and calculus really helps make our lives easier.

1.0.1 Nonlinear Curves

Have a nonconstant slope $y = f(x)$, and may be convex (slope increases as x increases) or concave (slope decreases as x increases).

As the slope changes depending where on the function you are looking, we are always going to be talking about the slope at a point on the curve.

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Slope at a point on a curve = slope of a line tangent to the curve at that point

Why calculus makes life easy The derivative of a function is simply the formula for the slope of the function. In general, it is a measure of the rate of change of a function's dependent variable for a change in the independent variable. Essentially with calculus we are analyzing the mathematics of motion and change, and differential calculus is used to find rates of change from a total.

For a function with one independent variable, $y = f(x)$, the derivative of y with respect to x is written as dy/dx or $f`(x)$ and is defined as the instantaneous change in y that results from an infinitesimal change in x.

In the language of calculus the symbol "*d*" is used for the derivative operator but primes "'" are also used as well as delta's Δ where:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right)$$

And the prime ' attached to the function of x, read as "f prime of x" indicates a first derivative. The notation

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right)$$

means the limit (or final value) that is approached by the ratio of the change in y to the change in x

$$\left(\frac{\Delta y}{\Delta x} \right)$$

,as the change in x approaches zero (or becomes very, very small).

NOTE 1: I use all three notational forms for derivatives interchangeably.

NOTE 2: for a function to be differentiable such that it has a derivative, it must be smooth (there cannot be any sharp points in a graph of the function) and it must be continuous (there cannot be any breaks in the graph of the function).

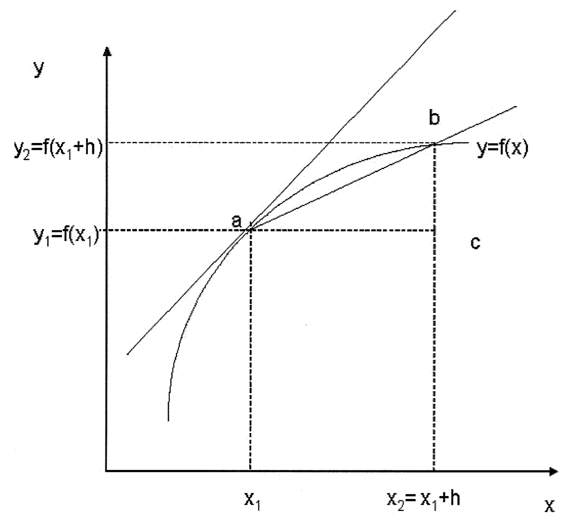

Figure 1: Tangent Slope

Graphically the derivative of a function at a point is given by the value of the slope of the line tangent to the curve $y = f(x)$ at that point. If the curve is linear, the slope is constant (remembering the equation of a straight (linear) line as $y = mx + b$ where m is the slope and b a constant). But for non-linear curves the slope depends where you are on the curve.

The slope of $y = f(x)$ is given at any point on the curve by the slope of the line tangent to the curve at that point. Consider the point a with coordinates (x_1, y_1). The derivative of the function $y = f(x)$ at the point a is equal to the slope of the line tangent to the curve at point a.

The slope of the tangent line is easy to calculate (remembering that the slope of a linear curve m=(change in dependent variable)/(change in independent variable)). For arbitrary points b and c, the slope of $ab = (bc/ac)$.

The goal of differential calculus is to find the slope of the tangent at a. We can approximate the slope of the function at a by finding the slope of the line segment between point a and any other point on $y = f(x)$, for example point d in the below

The slope of the line ad (where slope of $ad = (dc/ac)$) is less that the slope of the tangent at point a. But if we were able to move point d towards point a (or as x_2 approaches x_1 or as Δx approaches zero) the approximation of the

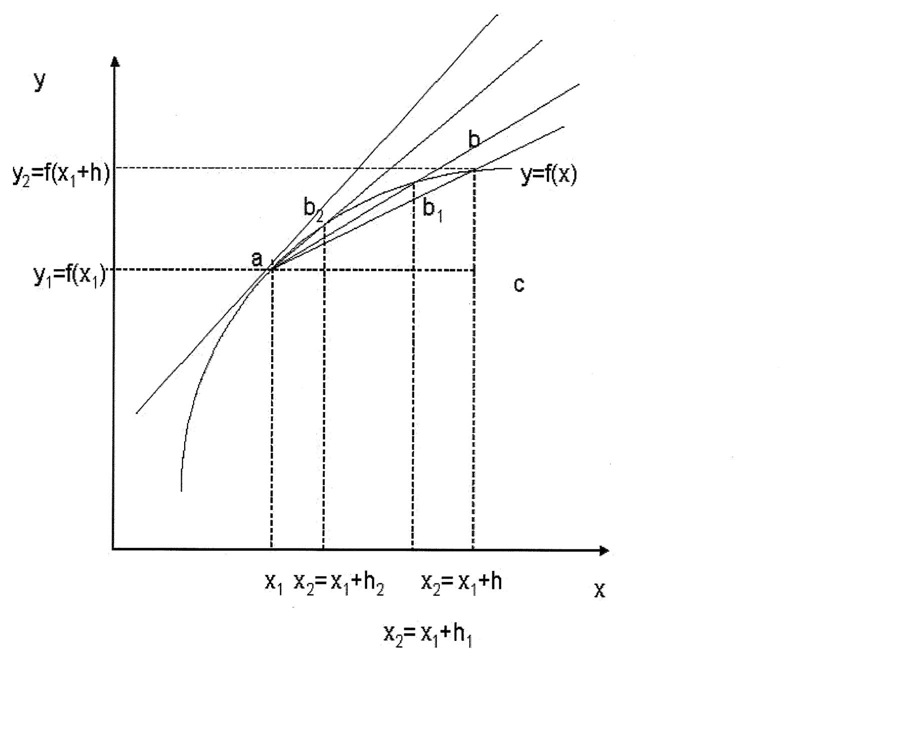

Figure 2: Approximate Slope

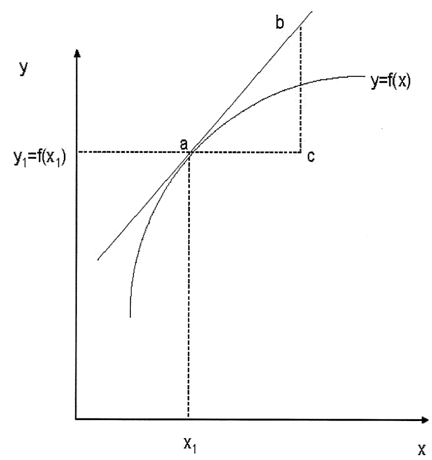

Figure 3: Difference Quotient

line segment gets closer to the slope of the tangent line. This is why we use the concept of limits where

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right)$$

just means what is the change in y for a change in x as the change in x (Δx) gets very close to zero – namely the slope of the tangent line.

To operationalize these concepts, refer to the below figure

Let h be an arbitrary "small" constant. For the two points a and b,

$$slope\ \ ab = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

So as h is reduced, the line segment will "pivot" up (in this case) towards the tangent.

Thus the slopes:

$$slope\ \ ab < slope\ \ ab_1 < slope\ \ ab_2$$

Or

$$\frac{f(x_1 + h) - f(x_1)}{h} < \frac{f(x_1 + h_1) - f(x_1)}{h_1} < \frac{f(x_1 + h_2) - f(x_1)}{h_2}$$

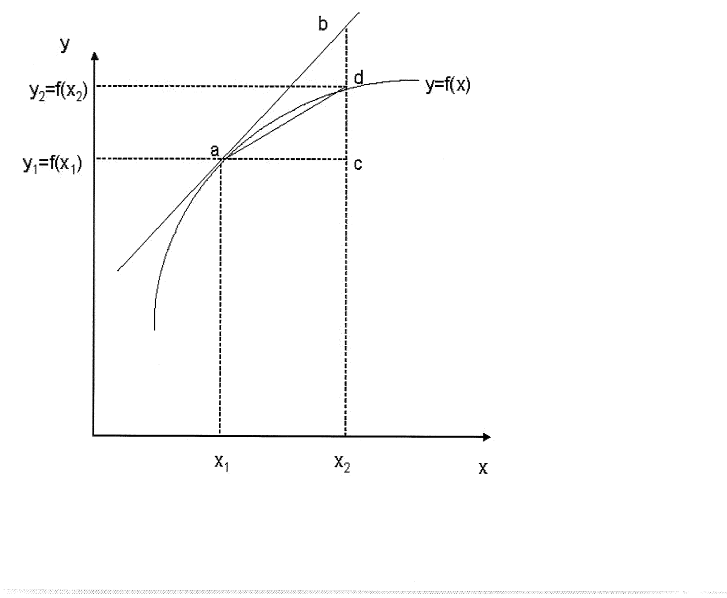

Figure 4: Reducing h

So as we keep making h smaller (towards zero) the slope found in this fashion will approach that of the tangent and therefore be the derivative of the function $y = f(x)$:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

And we can use this for any function. Let $f(x) = x^2$. Then point a would be given by (x, x^2) and point b by $(x+h, x^2+2xh+h^2)$. The slope of the function can then be found by using:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$

Which simplifies to:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h$$

So that we have:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} 2x + h = 2x$$

The slope therefore of $y = x^2$ is $dy/dx = 2x$.

Important Rules of Taking Derivatives

Constants – the derivative of a constant is always zero (as a horizontal line has slope =0)

Power Rule - Instead of following the above rather lengthy procedure we have a nice and easy rule for taking derivatives of general power functions in the form of $y = f(x) = ax^n$ where a and n are constants. The rule is:

$$y = ax^n \quad \frac{dy}{dx} = nax^{n-1}$$

Example: $f(x) = 3x^5$ -> $dy/dx = f`(x) = 15x^4$

There is another important reason why derivatives make life easier. We can use differentiation not only to find the slope at a point, but we can also use differentiation to find how that slope itself is changing, and this allows us to find minimum's and maximums (where slopes are zero). Differentiation not only allows us to find points with a zero slope (may be a min or a max) but we can also figure out which are mins and which are maxs by how the slope is changing in the vicinity of the extreme point.

e.g.

$$\begin{aligned}
y &= f(x) \\
&\quad \text{First order condition} \\
\frac{dy}{dx} &= 0 = f'(x) \\
&\quad \text{use the FOC to solve for } x^* \text{ the extreme point} \\
\frac{d^2y}{dx^2} &= f^{\prime'}(x) \\
\text{take another derivative} &: \\
\text{if } f^{\prime'}(x) &< 0 \text{ function is concave with a MAX} \\
\text{if } f^{\prime'}(x) &> 0 \text{ function is convex with a MIN}
\end{aligned}$$

The following site is also a great reference of calculating the slope of a curve with calculus: http://www.worsleyschool.net/science/files/curve/slope.html.